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Problem on Repeated limit.

Ex:→ let $f(x,y) = \frac{xy}{x^2+y^2}$ show that Repeated limits Exist at (0,0) and are equal but the Simultaneous limit does not Exist.

Here
$$\lim_{y\to 0}\lim_{x\to 0}\frac{xy}{x^2+y^2} = \lim_{y\to 0}\frac{0}{0+y^2} = 0$$

$$\lim_{x\to 0}\lim_{y\to 0}\frac{xy}{x^2+y^2} = \lim_{x\to 0}\frac{0}{x^2+0} = 0.$$

Thus two Repeated limits are equal

for Simultaneous limit

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2}$$ By putting $y = mx$

we have $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{x\to 0}\frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$

⇒ m may take up any value.

thus, here we see Repeated limits Exist and are equal, yet simultaneous limit does not exist.

Ex:→ let $f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}$ show that Repeated limits Exist and are equal but simultaneous limit does not Exist.

Soln :→ $$\lim_{y\to 0}\lim_{x\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}$$

$$\lim_{y\to 0}\frac{0}{y^2} = 0$$

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}$$

$$\lim_{x\to 0}\frac{0}{x^2} = 0$$

Thus two Repeated limits are equal.

Now for simultaneous limit

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2y^2}{x^2y^2+(x-y)^2}$$

By putting $y = x - mx^2$

we have

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = \lim_{x \to 0} \frac{x^2(x-mx^2)^2}{x^2(x-mx^2)^2 + (mx^2)^2}$$

$$= \lim_{x \to 0} \frac{\cancel{x^4}(1-mx)^2}{\cancel{x^4}[(1-mx)^2 + m^2]}$$

$$= \frac{1}{1+m^2}$$ which is different for different values of m

Therefore $\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y)$ does not exist.

Show that the simultaneous limit exists at the origin but the repeated limits do not exist for the function defined by

$$f(x,y) = x \sin\frac{1}{y} + y \sin\frac{1}{x}, \quad xy \neq 0$$

and $f(0,0) = 0$.

Sol^n: Let $\varepsilon > 0$ be given.

Here $\left| x \sin\frac{1}{y} + y \sin\frac{1}{x} - 0 \right| \leq |x| + |y|$

since $\sin\frac{1}{x} \leq 1$

$\sin\frac{1}{y} \leq 1$

$= 2\sqrt{x^2+y^2}$ $\quad \because |x| = \sqrt{x^2} < \sqrt{x^2+y^2}$

if $x^2 < \frac{\varepsilon^2}{4}, y^2 < \frac{\varepsilon^2}{4}$ i.e. if $|x| < \varepsilon/2 = \delta$

$|y| < \varepsilon/2 = \delta$

Thus given $\varepsilon > 0$ there exist $\delta > 0$ such that

$\left| x \sin\frac{1}{y} + y \sin\frac{1}{x} \right| < \varepsilon$ ~~that~~ whenever $|x| < \delta, |y| < \delta$

$\therefore \lim_{\substack{x \to 0 \\ y \to 0}} \left\{ x \sin\frac{1}{y} + y \sin\frac{1}{x} \right\} = 0$

Thus the simultaneous limit exists at the origin and its value is 0.

But $\lim_{x \to 0} f(x,y)$ and $\lim_{y \to 0} f(x,y)$ does not exist

Therefore the Repeated limits

$\lim_{y \to 0} \lim_{x \to 0} f(x,y)$ and $\lim_{x \to 0} \lim_{y \to 0} f(x,y)$ do not exist

Ex. → If $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$, $(x,y) \neq (0,0)$

and $f(0,0) = 0$

Show that $\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = 0$ [Do yourself]